Chandler's Theorem

By David Chandler, Corcoran High School, Corcoran CA 93212

David@DavidChandler.com

When I first taught geometry many years ago I was intrigued that a fair number of students would get the wrong answer on a test when shown two parallel lines and a transversal and asked whether two particular angles were equal[1]. One would think that apart from vocabulary, any third grader looking at the diagram could tell that all the acute angles are equal, all the obtuse angles are equal, and the acute and obtuse angles are different from each other (Fig. 1). Apparently the overlay of obscure vocabulary ("corresponding angles," "alternate interior angles," "alternate exterior angles," etc.) and "deep discussion" of the parallel postulate, etc. amounted to obfuscation, for some students, to the extent that they couldn't see the obvious in a practical situation.

The parallel postulate and related theorems tell why certain angles in the diagram are equal, but "just looking at it" gives the correct results in practical problems. The only time the method of "just looking at it" won't give the correct results is where the angles involved are sufficiently close to 90° that the acuteness or obtuseness of a given angle is not obvious. At that point one could fall back on rules about corresponding angles, alternate interior angles, etc. But another practical solution would be to exaggerate the angle of the transversal to make all acute and obtuse angles more obvious. This (heretical, right-brained) approach amounts to using a simple diagram as a mnemonic to

[1] I consciously avoid the "edu-speak" requirements of some geometry texts that insist we say angles are "congruent," when their "measures" are equal. Real-world engineers and physicists don't get bogged down in these pedantic issues, and beyond pointing out the distinctions at an appropriate time, students should not be burdened with them either.

concisely summarize a postulate, several theorems, and a lot of vocabulary. My position is that once an intuition has been "educated and clarified" by the formal mathematical process, that intuition can be useful as a practical problem-solving tool.

A student's geometry skills get a workout in introductory physics. Practical geometry is a prerequisite skill in constructing and labeling diagrams, free body or otherwise, used in physics problem solving. Using geometric theorems to relate a given angle to the needed angle to solve a problem can sometimes require a lengthy chain of reasoning. To simplify the process I introduce a little theorem (which I present to my physics classes, tongue-in-cheek, as "Chandler's Theorem"[2]).

> *Consider two lines forming an acute angle and a drawing consisting of any number of other lines in the same plane, all parallel or perpendicular to one or the other sides of the angle. All acute angles in the drawing are equal to either the given angle or its complement.*

This is a very useful theorem for immediately seeing angle relationships in complex diagrams *as long as the original angle and its complement are easily distinguishable*.

Consider a car going around a banked curve traveling in a horizontal plane (Fig. 2). Let's assume the car is going slower than the optimal speed and is kept from sliding down the embankment by friction. The problem can be solved by breaking the forces into horizontal and vertical components, summing the horizontal and vertical components separately, equating the net vertical force to zero and equating the net horizontal force, to

[2] I also fill gaps in the S.I. unit system with the names of students in my classes. For example, the neglected S.I. unit of momentum (kg m/s) might, for some particular school year, be a "Gomez."

$\frac{mv^2}{r}$. But first the student must draw a free body diagram and correctly identify all the forces and their components. Gravity is vertical, the normal force is perpendicular to the road, and the friction is parallel to the road. The components of the forces are all horizontal or vertical. Note that all lines in the drawing are either parallel or perpendicular to the road or the horizontal line. The diagram thus satisfies the criteria of Chandler's Theorem. As long as the diagram is drawn with the bank angle significantly different from 45° (regardless of its actual value of the angle stated in the problem) a student will be able to distinguish at a glance the "little" acute angles from the "big" acute angles. If the bank angle is 23°, there is no danger that some angle in the diagram might be 24.5°. All acute angles in the drawing are either 23° or 67°. "Just looking at it" gives the correct results. Having a theorem to justify the intuition removes the "guilt!" Even if the student (or the teacher) feels the need to derive the angles one step at a time, having a good idea of the answer is a good starting point for solving the problem.

Proof of "Chandler's Theorem"

To prove the theorem we start with a diagram that meets the criteria of the theorem. On a case-by case basis we consider 1) the possible angles formed where any other line in the diagram meets one of the sides of the original angle, and 2) the possible angles formed where any two other lines in the drawing meet each other. Listing the cases is the tricky part. The proofs, in each case, are standard one or two step geometry exercises.

Consider lines *j* and *k* meeting to form an acute angle θ, and any other two lines *m* and *n* in the drawing. (Let the complement of θ be designated ϕ.) All angles formed by these lines (other than the original angle) occur either where *m* meets *n* or where one of these lines meets *j* or *k*. The following cases are to be proved. (See Fig. 3)

Case 1. If $m \| j$, the acute angles where *m* meets *k* equal θ.

Case 2a. If $m \perp j$, and *m*, *j*, and *k* are not coincident, the acute angles where *m* meets *k* equal ϕ.

Case 2b. If $m \perp j$, and *m*, *j*, and *k* are coincident, all the acute angles at the point of coincidence equal θ or ϕ.

Apart from re-labeling, these exhaust the intersections of the lines with the sides of the original angle. All that remain to be considered are the intersections of lines *m* and *n*.

Case 3. If $m \| j$ and $n \perp j$, *m* and *n* meet at right angles.

Case 4. If $m \| j$ and $n \| k$, the acute angles where *m* meets *n* equal θ.

Case 5a. If $m \| j$ and $n \perp k$, and no three lines are coincident, the acute angles where *m* meets *n* equal ϕ.

Case 5b. If $m \| j$ and $n \perp k$, and *m*, *n*, and *k* are coincident, all the acute angles at the point of coincidence equal θ or ϕ.

Case 6a. If $m \perp j$ and $n \perp k$, and no three lines are coincident, the acute angles where *m* meets *n* equal θ.

Case 6b. If $m \perp j$ and $n \perp k$, and *m*, *n*, and *k* are coincident, all the acute angles at the point of coincidence equal θ or ϕ.

Case 6c. If $m \perp j$ and $n \perp$k, and all four lines are coincident, all the acute angles at the point of coincidence equal θ or ϕ.

Apart from re-labeling, these cases are exhaustive. (The proofs of the individual cases are left as an exercise for the reader.) Once it is seen that θ and ϕ are the only acute angles that can occur between any pair of lines in the drawing, the theorem is proved.

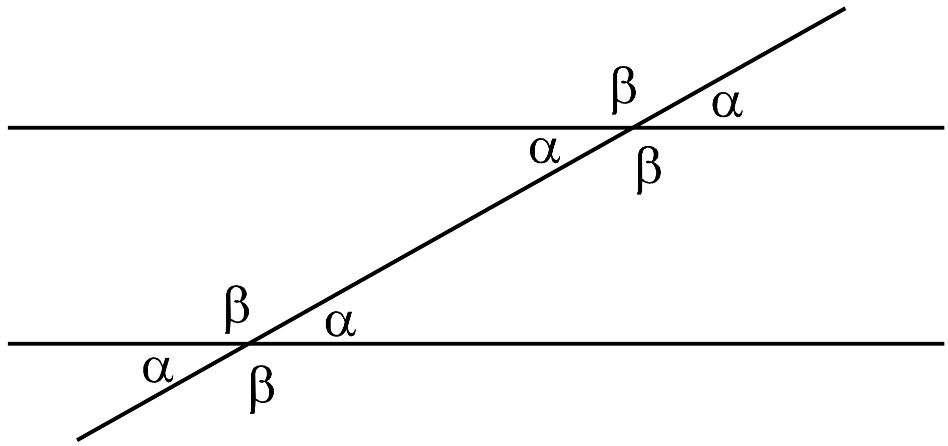


Figure 1

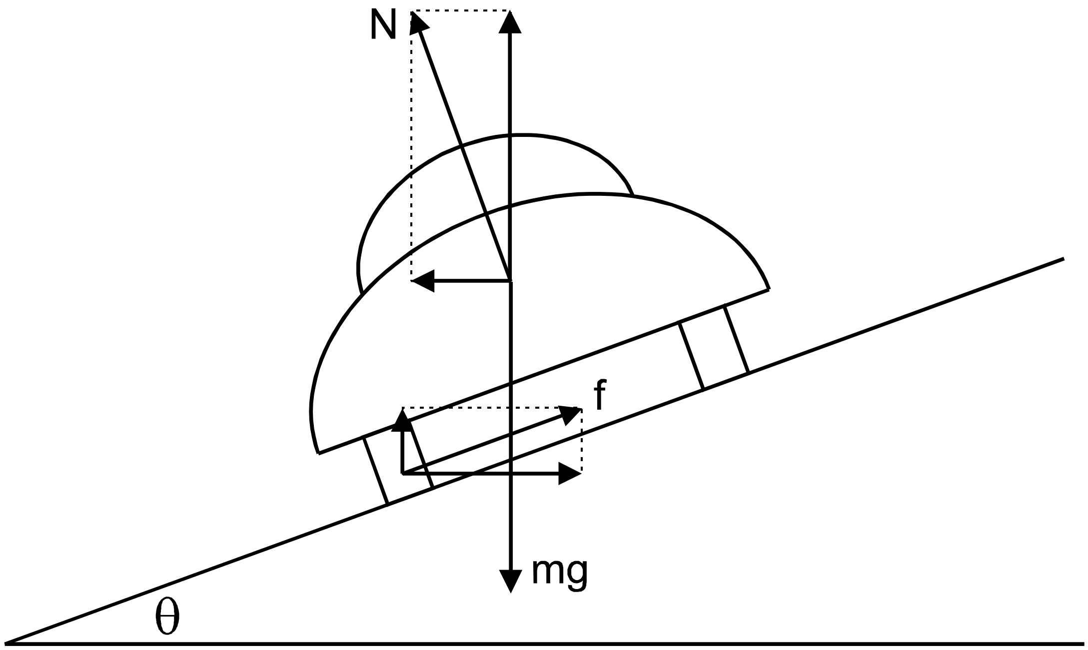
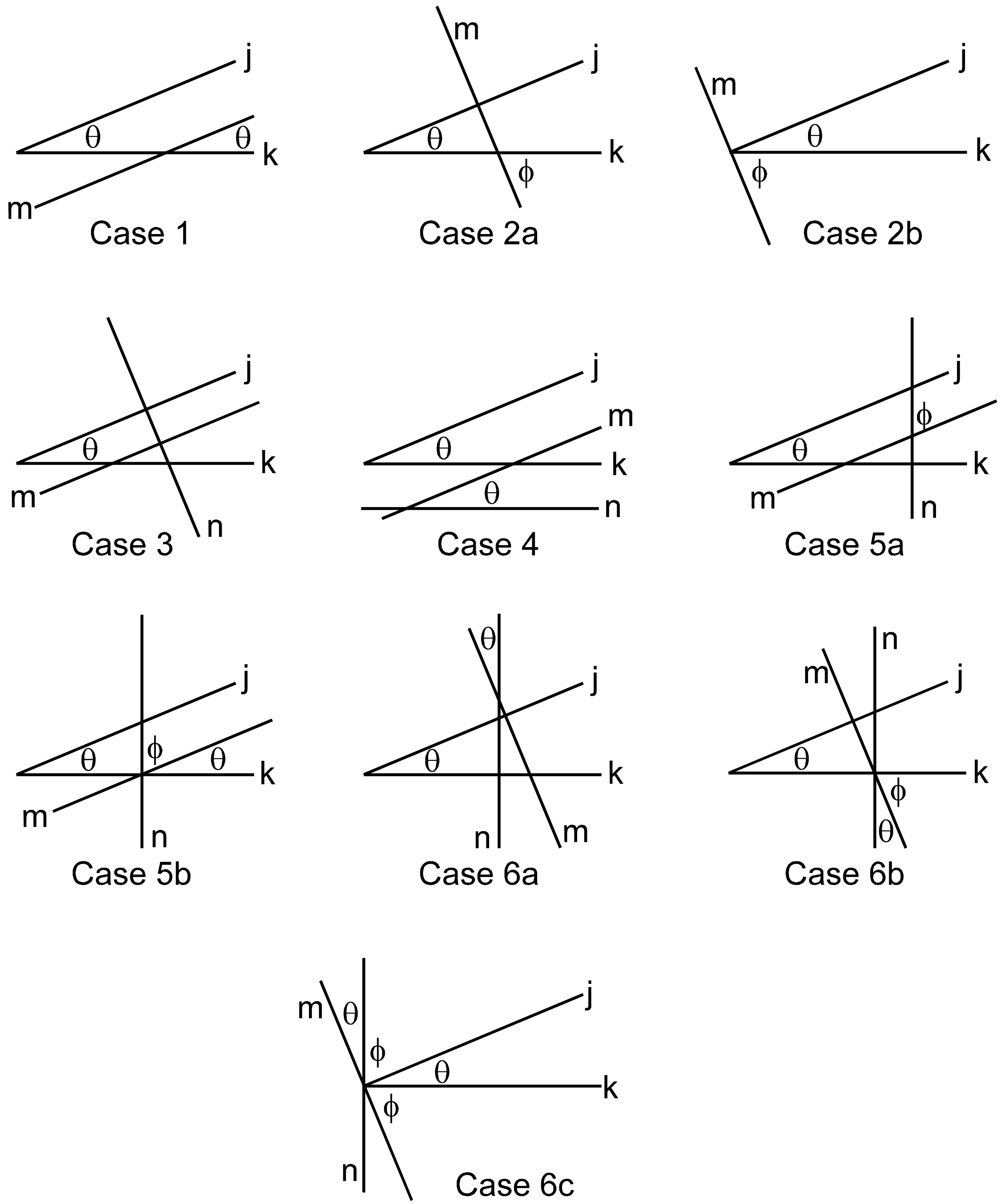


Figure 2


j
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Case 1
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Case 4
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Case 6a
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Case 6b
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Case 6c

Figure 3