Motivating Imaginary Numbers

By David Chandler

Historically, imaginary numbers were first taken seriously in the study of cubic equations. The history of cubic equations in the 15th and 16th centuries is "complex" but a starting point would be to research the names Scipione del Ferro, Niccolo Tartaglia, Niccolo Tartaglia, Gerolamo Cardano, and Rafael Bombelli. The connection with imaginary numbers is that cubic equations known to have three real roots can be constructed by multiplying three linear binomials. Yet Tartaglia's method of solution leads to intermediate expressions involving the square roots of negative numbers. If one throws up his or her hands and declares such numbers to be nonsense, one is blocked from finding the solutions. However taking such numbers seriously, and assuming they follow the ordinary rules of arithmetic, leads to tangible results. Once imaginary numbers were seen to be useful in one context, they were explored and developed in their own right.

One of the quandaries faced by mathematics teachers is how to motivate the discussion of imaginary numbers. Why should students be expected to take these numbers seriously when they have never experienced a need for them and have no conceptual hook on which to hang them?

A common strategy in textbooks today is to appeal to the concept of closure. Imaginary numbers allow all quadratic equations to have solutions. In my experience students do not find this explanation to be satisfying. What's wrong, after all, with some quadratic equations not having solutions? For instance when throwing spit wads at the ceiling one might ask at what time the spit wad would make contact, given the height and launch velocity. Spit wads that are launched too slowly would simply not hit the ceiling. There would be no solution, and none would be expected. Insisting that even such equations must have solutions seems as bizarre to students as simply asserting that imaginary numbers exist.

It wasn't abstract ideas like closure that led mathematicians to accept imaginary numbers. They were accepted because they were found to be useful. One might be tempted to introduce the solution of cubic equations into the curriculum, but I think most of us would agree this is a little ambitious for an Algebra II or Pre Calculus class.

I propose a different problem that has the same pedagogical value. It is a method that successfully solves a certain class of problems, but fails on some problems unless imaginary numbers are allowed. Once imaginary numbers are allowed, and we continue to do arithmetic with them, real, tangible solutions are once again attainable. In a sense, we are *simulating* the scenario that led to the original acceptance of imaginary numbers.

Consider the problem of finding the axis of symmetry of a quadratic function. (If students wonder why we would ever want to know how to do that, suggest using it to find when and where a projectile reaches the maximum height on a parabolic trajectory.) One conceptually simple approach is to find the x-intercepts by setting y=0 and finding the roots of the corresponding quadratic equation. We can then average the x-intercepts to find the midpoint. The vertical line through the midpoint will be the axis of symmetry.

Start with a concrete example with real solutions. For example, find the axis of symmetry for the parabola $y=x^2+3x-10$. Setting y=0 and evaluating the quadratic formula, we see the x-intercepts occur at 2 and -5. The midpoint is at x=-3/2. This is an easy success which can be reinforced by graphing.

Now let's try another example: $y=x^2+3x+10$. This time the quadratic formula yields

$$x=\frac{-3\pm\sqrt{-31}}{2} \quad ,$$

so we have a problem with no real solutions. We could, at this stage, give up on the approach of averaging the x-intercepts and go hunting for a different method altogether that would not run into this problem, but “averaging the roots” is such an appealing, easily remembered, and easily implemented solution that it would be a shame to dump it if it could be made to work.

If instead we allow the $\sqrt{-31}$ to be treated as a valid number, the midpoint of the two x-intercepts can be found:

$$\frac{1}{2}\left(\frac{-3+\sqrt{-31}}{2}+\frac{-3-\sqrt{-31}}{2}\right)=\frac{1}{2}\left(\frac{-3-3}{2}\right)=-\frac{3}{2} \quad .$$

The imaginary numbers cancel out and we achieve our goal. The key point is they only "cancel out" if we decide it is OK to do arithmetic with them. That in itself gives them "number status," in some sense.

Once we agree that we are willing to do arithmetic with square roots of negative numbers, we can do the problem in general, not caring whether the intermediate expressions are real or not. The axis of symmetry of the general quadratic function is thus

$$\frac{1}{2}\left(\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}\right)=\frac{1}{2}\left(\frac{-b-b}{2a}\right)=-\frac{b}{2a}$$

It is tempting for students to do this derivation without worrying about whether the discriminant is positive or negative, but it should be pointed out to them that this simple formulaic result depends on the assumption that we can do arithmetic with square roots of negative numbers.

Once imaginary numbers are introduced, the complex plane should be introduced as soon as possible so students will have a place to "put" these numbers.

Conclusion

The study of complex numbers does not always result in solutions where the imaginary values cancel out, but as a stepping stone to a general conception of complex numbers, the line of symmetry problem is appealing, easy to visualize, and leads to a nice, clean result. Additionally, it is a historical analogue: it can play the same role for students today that cubic equations played for 15^{th} and 16^{th} century mathematicians.