NOTE

Newton's Second Law for Systems with Variable Mass

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The usual model used to illustrate variable-mass problems is a rocket. A rocket is a familiar enough motivating example, but it is a poor example to use in the actual derivation because it introduces a confusing mixture of positive and negative signs. In the rocket problem, $\frac{dm}{dt}$ is negative, the velocity v_{rel} of the exhaust is negative and measured relative to the rocket, whereas the thrust is positive. The derivation would be a lot cleaner if an example were used with all positive terms.

My model, for derivation purposes, is a wagon being filled and propelled by water coming from a hose (see Fig. 1). The system consists of the wagon and the water it is carrying at the moment. (I point out at the start that only the horizontal component of momentum is under consideration.) The water from the hose adds to the mass of the system at the positive rate $\frac{dm}{dt}$. The entering water has its own momentum, so as it crosses the system boundary it transfers momentum into the system at the rate $\frac{dp_{\mathrm{w}}}{dt}$. In a calculus-based course, Newton's second law would already have been introduced in the form $F = \frac{dp}{dt}$, so $\frac{dp_{\mathrm{w}}}{dt}$ is easily recognizable as a force acting on the system. There may be other forces acting besides the water flow, such as a rope pulling on the wagon, gravity pulling the wagon down a hill, friction, etc. We combine all such forces together under the heading F_{EXT}. The *net* force acting on the system is therefore $F_{\mathrm{EXT}} + \frac{dp_{\mathrm{w}}}{dt}$, so Newton's second law becomes $F_{\mathrm{EXT}} + \frac{dp_{\mathrm{w}}}{dt} = \frac{dp_{\mathrm{sys}}}{dt}$. If the speed of the in-flowing water is u (a constant speed measured relative to the ground) and the instantaneous speed of the wagon is v, then expanding the two $\frac{dp}{dt}$ terms yields:

$$F_{\mathrm{EXT}} + u\frac{dm}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}.$$

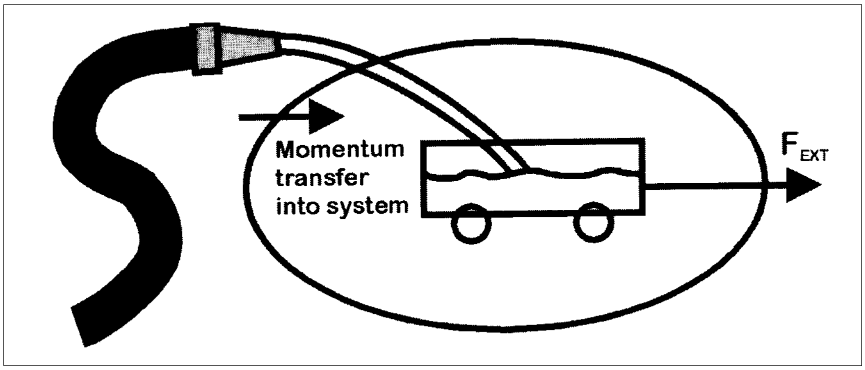

Fig. 1. Momentum is transferred into the system as mass is added. Other external forces may also act on the system.

(The expansion of $\frac{dp_{\mathrm{w}}}{dt}$ on the left produces only a single term because the speed u of the water is taken to be constant.) At this point it is worth pausing to notice that $u\frac{dm}{dt}$ appears on the left side of the equation as a force on the system, whereas $v\frac{dm}{dt}$ appears on the right side of the equation as part of the change of momenum of the system. This is usually one of the sticking points for students in traditional derivations, but in this derivation the terms arrive at their correct places in the equation very naturally.

The equation $F_{\mathrm{EXT}} + u\frac{dm}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$ is useful "as is" for problems where all velocities are measured with respect to the ground. If relative velocities are used, as in rocket problems, the two $\frac{dm}{dt}$ terms can be combined, $F_{\mathrm{EXT}} + (u - v)\frac{dm}{dt} = m\frac{dv}{dt}$, and $v_{\mathrm{rel}} = u - v$ can be introduced, yielding $F_{\mathrm{EXT}} + v_{\mathrm{rel}}\frac{dm}{dt} = m\frac{dv}{dt}$. For rocket problems, the second term on the left is interpreted as thrust. Since for rockets v_{rel} is negative and $\frac{dm}{dt}$ is negative, the thrust is positive.